DIFFERENZIALE SECONDO:

$$
\begin{aligned}
d^2 f = {} & \underbrace{\frac{\partial^2 f}{\partial x_1^2} dx_1^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 dx_2 + \dots + \frac{\partial^2 f}{\partial x_1 \partial x_n} dx_1 dx_n} \\
& + \underbrace{\frac{\partial^2 f}{\partial x_2 \partial x_1} dx_2 dx_1 + \frac{\partial^2 f}{\partial x_2^2} dx_2^2 + \dots + \frac{\partial^2 f}{\partial x_2 \partial x_n} dx_2 dx_n} \\
& + \dots + \underbrace{\frac{\partial^2 f}{\partial x_n \partial x_1} dx_n dx_1 + \frac{\partial^2 f}{\partial x_n \partial x_2} dx_n dx_2 + \dots + \frac{\partial^2 f}{\partial x_n^2} dx_n^2}
\end{aligned}
$$

$$
d^2 f = dx^T \underbrace{\nabla^2 f(x^0)}_{\text{HESSIANA}} dx
\qquad
\begin{bmatrix} dx_1 & dx_2 & \cdots & dx_n \end{bmatrix}
\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & & \\ & \ddots & \\ & & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}
\begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}
$$

per f a 2 variabili :

$$\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$$

$$d^2 f = \frac{\partial^2 f}{\partial x_1^2} dx_1^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 dx_2 +$$

$$+ \frac{\partial^2 f}{\partial x_2^2} dx_2^2 + \frac{\partial^2 f}{\partial x_2 \partial x_1} dx_2 dx_1$$

$=$

$f \in C^2$

$$d^2 f = \frac{\partial^2 f}{\partial x_1^2} dx_1^2 + \frac{\partial^2 f}{\partial x_2^2} dx_2^2 + 2 \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 dx_2$$

$$d^2 f = \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

$$d^2f = \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} \, // & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

ex: $f(x_1, x_2) = -x_1^2 + \ln x_1 x_2$

$$d^2f = \frac{\partial^2 f}{\partial x_1^2} dx_1^2 + \frac{\partial^2 f}{\partial x_2^2} dx_2^2 + 2 \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 dx_2$$

$\log = \ln$

Log

$\log_2(x)$

PUNTO $(1, 1)$

$(x_1 - 1) = dx_1$

$(x_2 - 1) = dx_2$

$$-x_1^2 + \ln x_1 x_2 = f(x_1, x_2)$$

$$\frac{\partial f}{\partial x_1} = -2x_1 + \frac{\cancel{x_2}}{x_1 \cancel{x_2}} = -2x_1 + \frac{1}{x_1}$$

$$\frac{\partial f}{\partial x_2} = \frac{\cancel{x_1}}{\cancel{x_1} x_2} = \frac{1}{x_2}$$

$$\frac{\partial^2 f}{\partial x_1^2} = -2 - \frac{1}{x_1^2} \Rightarrow -2-1$$

$$\frac{\partial^2 f}{\partial x_2^2} = -\frac{1}{x_2^2} \Rightarrow -1$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

$$H = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$$

$H(1,1)$

$dx_1 = (x_1 - 1) \quad dx_2 = (x_2 - 1)$

$$d^2 f = \begin{bmatrix} x_1-1 , & x_2-1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1-1 \\ x_2-1 \end{bmatrix}$$

$$d^2 f = \frac{\partial^2 f}{\partial x_1^2} dx_1^2 + \frac{\partial^2 f}{\partial x_2^2} dx_2^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 dx_2$$

$$d^2 f = -3 (x_1-1)^2 + (-1)(x_2-1)^2 + \underbrace{0 \cdot (x_1-1)(x_2-1)}_{= 0}$$

$$= -3 (x_1-1)^2 - (x_2-1)^2$$

POLINOMIO DI TAYLOR (X F. A 1 VARIABILE)

$f : (a,b) \to \mathbb{R} \qquad x_0 \in (a,b)$

è possibile approssimare f vicino a x_0 con un polinomio del tipo $P_n(x, x_0) = a_0 + a_1(x - x_0) + \dots + a_n(x - x_0)^n$

TEOREMA :

Dato $f : (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$, $f \in C^n(a,b)$,

$\to \exists!\ P_n(x, x_0)$ tale che : $\displaystyle\lim_{x \to x_0} \frac{f(x) - P_n(x, x_0)}{(x - x_0)^n} = 0$.

$$P_n(x, x_0) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots + \frac{1}{n!} f^{n}(x_0)(x - x_0)^n$$

si chiama il polinomio di Taylor di grado n
e punto iniziale x_0

POLINOMIO DI TAYLOR x FUNZIONE A + VARIABILI:

Sia A un insieme aperto $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$

$f \in C^m(A)$ punto x^0 $\quad x^0+h$

$$f(x^0+h) = f(x^0) + df(x^0) + \frac{1}{2} d^2 f(x^0) + \dots + \frac{1}{m!} d^m f(x^0)$$

$$h^T \nabla f(x^0) \qquad\qquad h^T \nabla^2 f(x^0)\, h$$

$$df \qquad\qquad\qquad\qquad d^2 f$$

FUNZIONE COBB-DOUGLAS:

$$F(x,y) = x^{1/4}\, y^{3/4}$$

PUNTO (1 1) iniziale

PUNTO (1,1 0,9)

$$\frac{\partial f}{\partial x} = \frac{1}{4} x^{-3/4} y^{3/4} \to \frac{1}{4}$$

$$\frac{\partial f}{\partial y} = x^{1/4} \frac{3}{4} y^{-1/4} \to \frac{3}{4}$$

$$h = [0,1 \ , \ -0,1]$$

$$f(1,1)$$

$$[0,1 \ , \ -0,1]$$

$$\left[\frac{\partial f}{\partial x}(1,1) \ , \ \frac{\partial f}{\partial y}(1,1)\right]$$

$$f(x^0+h) = f(x^0) + \underbrace{h^T \nabla f(x^0)}$$

1° ORDINE

+ RESTO

$$f(1{,}1\,,\,0{,}9) = f(1,1) + \begin{bmatrix} 0{,}1 & -0{,}1 \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$$

$$f(x^0+h) = f(x^0) + h^T \nabla f(x_0)$$

$$f\Big((1,1)+(\underset{0{,}1 \quad -0{,}1}{h})\Big)$$

$$\underline{f(1{,}1\,,\,0{,}9)} = f(1,1) + \overbrace{0{,}1 \cdot 1/4} - \overbrace{0{,}1 \cdot 3/4} = 0{,}95$$

$$= 1 + 0{,}1 \cdot \frac{\partial f}{\partial x_1} - 0{,}1 \cdot \frac{\partial f}{\partial x_2} =$$

il valore reale in $(1{,}1\ ,\ 0{,}9)$ è

$$f(1{,}1,\ 0{,}9) = 1{,}1^{1/4} \cdot 0{,}9^{3/4} = 0{,}9463$$

il valore approssimato con polinomio Taylor

1° ORDINE è $= 0{,}95$

$$f(1{,}1\ ,\ 0{,}9) = \underbrace{f(1,1) + 0{,}1 \cdot \frac{1}{4} + (-0{,}1) \cdot \frac{3}{4}}_{1° \text{ ORDINE} = 0{,}95} + \underbrace{\frac{1}{2} d^2 f}$$

$$f(1{,}1,\ 0{,}9) = 0{,}95 + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2} dx^2 + \frac{\partial^2 f}{\partial y} dy^2 + 2 \cdot \frac{\partial^2 f}{\partial x \partial y} dx\, dy\right]$$

$$\frac{\partial f}{\partial x} = \frac{1}{4} x^{-3/4} y^{3/4}$$

$$\frac{\partial f}{\partial y} = x^{1/4} \frac{3}{4} y^{-1/4}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{4} \cdot \left(-\frac{3}{4} x^{-\frac{7}{4}}\right) y^{3/4} \longrightarrow -\frac{3}{16}$$

$$\frac{\partial^2 f}{\partial y^2} = x^{1/4} \cdot \frac{3}{4} \left(-\frac{1}{4}\right) y^{-\frac{5}{4}} \longrightarrow -\frac{3}{16}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{4} x^{-3/4} \frac{3}{4} y^{-1/4} \longrightarrow \frac{3}{16}$$

$$f(1{,}1,\ 0{,}9) = 0{,}95 + \frac{1}{2}\left[-\frac{3}{16}(0{,}1)^2 + \left(-\frac{3}{16}\right)(-0{,}1)^2 + 2\left(\frac{3}{16}\right)(0{,}1)(-0{,}1)\right]$$

APPROSSIMAZIONE CON POLINOMIO 2° ORDINE

↳ $= 0{,}94625$ più vicino al valore reale $0{,}9463$

$$h = [0{,}1\ ,\ -0{,}1]$$